STAT 2593

Lecture 017 - Cumulative Distribution Functions and Expected Values

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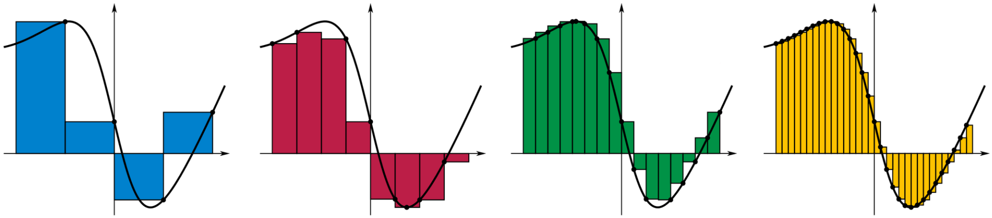
Cumulative Distribution Functions and Expected Values

Learning Objectives

1. Understand how the CDF and PDF are related.
2. Understand how the concepts from discrete random variables (expectation, variance, etc.) translate to continuous random variables.
3. Understand how percentiles relate to continuous distributions.



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  - We get $E[h(X)] = \int_{-\infty}^{\infty} h(x)f(x)dx$ and $\text{var}(X) = \int_{-\infty}^{\infty}(x - E[X])^2 f(x)dx$.
- We saw in deriving the PDF that the CDF is also given through integration as
$$F_X(x) = \int_{-\infty}^{x} f(t)dt.$$

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  - In general, we **do not** need to be careful about inequalities for continuous random variables.

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- Finding $\eta(0.5)$ gives the median (50-th percentile).

Summary

- Generally, summations can be replaced by integrals and have results maintained.
- The expectation, variance, and CDF are all defined analogously to the discrete case.
- Percentiles can be analytically computed for continuous distributions by inverting the CDF.